ON THE STEADY REGIME OF AN HEREDITARILY ELASTIC OSCILLATOR

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The weakly singular functions which are used as kernels of the Boltzmann–Volterra integral equations in the solution of statistical problems of the hereditary elasticity theory [1] have been widely used in the solution of dynamic problems as well [2]. An entire series of studies [3-5] has been made of the behavior of the one-dimensional oscillator alone. In the present paper we study the previously unknown characteristics of the steady-state regime of a one-dimensional oscillator whose hereditary characteristics are described by functions having an integrable singularity of the Abel type.

1. By virtue of the known connection between the relaxation R(t) and aftereffect kernels K(t) the equation of motion of the hereditarily elastic oscillator can be written in the equivalent forms

$$x^{\cdot\cdot} + \omega_\infty^2 x - (\omega_\infty^2 - \omega_0^2)\int_0^\infty R(t')\, x(t-t')\, dt' = p \sin \omega t \tag{1.1}$$

$$x^{\cdot\cdot} + \omega_\infty^2 x + \nu_\sigma \int_0^\infty K(t')\, x^{\cdot\cdot}(t-t')\, dt' = p\left[\sin \omega t + \nu_\sigma \int_0^\infty K(t') \sin \omega (t-t')\, dt'\right]$$
$$\nu_\sigma = (E_\infty - E_0)\, E_0^{-1}, \tag{1.2}$$

Here x is the coordinate, the overdot denotes derivative with respect to time, p is the per-unit-mass amplitude of the external monoharmonic force acting with the frequency ω, and the relaxed E_0 and unrelaxed E_∞ values of the elastic modulus define the corresponding elastic oscillation eigenfrequencies ω_0 and ω_∞.

For the stationary solution

$$x = X \sin (\omega t - \varphi_1) \tag{1.3}$$

equations (1.1) and (1.2) are written in the form of the elastic-viscous analogy

$$(\omega_0^2 A - \omega^2)x + \omega_0^2 \omega^{-1} B x^{\cdot} = p \sin \omega t, \quad (\omega_\infty^2 - \omega^2 C)x + \omega D x^{\cdot} = P \sin (\omega t - \varphi_2) \tag{1.4}$$

$$P \equiv p\,(C^2 + D^2)^{1/2}, \quad \operatorname{tg} \varphi_2 = D C^{-1} \tag{1.5}$$

The amplitude X and the phase φ_1 are defined by the expressions

$$a = X p^{-1} = [(\omega_0^2 A - \omega^2)^2 + \omega_0^4 B^2]^{-1/2} = P p^{-1} [(\omega_\infty^2 - \omega^2 C)^2 + \omega^4 D^2]^{-1/2} \tag{1.6}$$

$$\operatorname{tg} \varphi_1 = B\,[A - (\omega / \omega_0)^2]^{-1} = D\,[C - (P\omega / p\omega_\infty)^2]^{-1} \tag{1.7}$$

Here

$$A \equiv 1 + \nu_\sigma \left(1 - \int_0^\infty R(t) \cos \omega t\, dt\right), \quad B \equiv \nu_\sigma \int_0^\infty R(t) \sin \omega t\, dt, \quad C \equiv 1 + \nu_\sigma \int_0^\infty K(t) \cos \omega t\, dt, \quad D \equiv \nu_\sigma \int_0^\infty K(t) \sin \omega t\, dt \tag{1.8}$$

It is also not difficult to find the inverse Q^{-1} of the system quality, which is taken as the measure of the internal friction

$$Q^{-1} = \frac{1}{2\pi} \frac{\Delta W}{W} \tag{1.9}$$

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Here ΔW is the energy dissipated in the course of one cycle, and W is the maximal value of the energy in a given cycle.

In accordance with (1.4) ΔW is calculated from the formula

$$\Delta W = p \int_0^{2\pi/\omega} x^{\cdot} \sin \omega t dt = \pi p X \sin \varphi_1 \quad (1.10)$$

or is found as the area of the hysteresis loop

$$\Delta W = \pi X^2 \omega_0^2 B = \pi X^2 \omega_\infty^2 D (C^2 + D^2)^{-1} \quad (1.11)$$

It is easy to see the equivalence of (1.10) and (1.11) after substituting therein the values of X and φ_1 from (1.6) and (1.7). The maximal value of the energy is defined by the expression

$$W = {}^1/_2 X^2 \omega_0^2 A = {}^1/_2 X^2 \omega_\infty^2 C (C^2 + D^2)^{-1} \quad (1.12)$$

As a result we obtain the formula for the internal friction

$$Q^{-1} = BA^{-1} = DC^{-1} = \operatorname{tg} \varphi_2 \quad (1.13)$$

We note that in the quasistatic case, i.e., when the inertial properties of the system can be neglected, the tangent of the phase shift angle (1.7) becomes the tangent of the mechanical loss angle, which coincides identically with Q^{-1}(1.13).

More detailed information on the calculation of the oscillations of elastically hereditary systems is given in [6].

2. As the first example we shall examine the very simple weakly singular Abel kernel, which we take as the aftereffect kernel

$$K(t) = t^{\gamma-1} / \tau_\sigma^\gamma \Gamma(\gamma) \qquad (0 < \gamma \leqslant 1) \quad (2.1)$$

Here τ_σ is the retardation time. A similar kernel, only without the gamma function $\Gamma(\gamma)$, was used by Duffing [7] for analysis of the static creep curves of belts and other materials. The direct solution of the dynamic problem with the Duffing kernel leads only to the appearance of the gamma function in the final formulas, while the dependence on the frequency ω and the retardation time τ_σ (relaxation time τ_ε) remains the same.

The resolvent of the kernel (2.1) – the relaxation kernel – is defined by the Rabotnov fractional-exponential function [8]

$$R(t) = \tau_\varepsilon^{-\gamma} \vartheta_\gamma(-\nu, \tau, t), \quad \vartheta_\gamma \equiv \sum_{n=0}^{\infty} \left(-\frac{\nu}{\tau^\gamma}\right)^n \frac{t^{\gamma(n+1)-1}}{\Gamma[\gamma(n+1)]}, \quad \frac{E_0}{E_\infty} = \left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^\gamma \quad (2.2)$$

It is not difficult to calculate the sine and cosine of the Fourier transformant functions (2.1) and (2.2) and then the quantities

$$A = \frac{(1+\nu_\sigma)(\varkappa^\gamma \nu^{-1} + \cos\psi)}{\varkappa^\gamma \nu^{-1} + \varkappa^{-\gamma}\nu + 2\cos\psi}, \qquad B = \frac{(1+\nu_\sigma)\sin\psi}{\varkappa^\gamma \nu^{-1} + \varkappa^{-\gamma}\nu + 2\cos\psi} \quad (2.3)$$

$$C = 1 + \nu\varkappa^{-\gamma}\cos\psi, \quad D = \nu\varkappa^{-\gamma}\sin\psi, \quad \varkappa \equiv \omega\tau, \quad \psi = {}^1/_2 \pi\gamma \quad (2.4)$$

Substituting these values into (1.6) and (1.7), we find respectively the amplitude, tangent of the phase shift angle, and internal friction

$$a = \left(\frac{\varkappa^\gamma \nu^{-1} + \varkappa^{-\gamma}\nu + 2\cos\psi}{\Omega_\infty^2 \varkappa^\gamma \nu^{-1} + \omega^4 \nu \varkappa^{-\gamma} - 2\omega^2 \Omega_\infty}\right)^{1/2}, \qquad \Omega_\infty \equiv \omega_\infty^2 - \omega^2 \quad (2.5)$$

$$\operatorname{tg}\varphi_1 = [\Omega_\infty(\cos\psi + \varkappa^\gamma\nu^{-1}) - \omega^2(\cos\psi + \varkappa^{-\gamma}\nu)]^{-1} \omega_\infty^2 \sin\psi \quad (2.6)$$

$$Q^{-1} = \operatorname{tg}\varphi_2 = (\cos\psi + \varkappa^\gamma \nu^{-1})^{-1} \sin\psi \quad (2.7)$$

In (2.2)-(2.7) and hereafter, where the quantities τ and ν appear without indices it is assumed that $\tau = \tau_\sigma$ for $\nu = \nu_\sigma$ and $\tau = \tau_\varepsilon$ for $\nu = \nu_\varepsilon = \Delta E/E_\infty$.

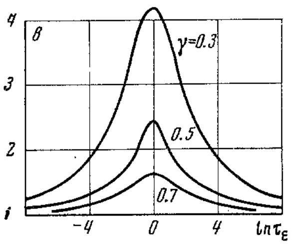

Fig. 1

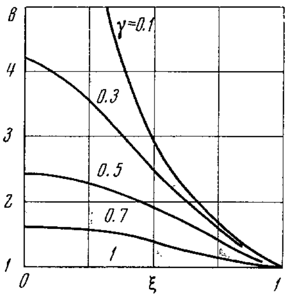

Fig. 2

For $\gamma = 1$ use of the kernels (2.1) and (2.2) in the Boltzmann–Volterra integral relations leads to the Maxwell rheological model to within the constant ν_ε ($\nu_\varepsilon = 1$ if $E_0 = 0$). In this case the dynamic formulas (2.5)-(2.7) correspond to the Maxwell model, for which at the frequency $\omega = \omega_* = \omega_\infty/\sqrt{2}$ all the resonant amplitudes intersect at a single point, regardless of τ, reaching at this point the values $a_*^{(0)} = 2\omega_\infty^{-2}$ [9]. For the kernels (2.1) and (2.2) at the frequency $\omega = \omega_*$ only the resonant amplitudes corresponding to $\tau = 0$ and $\tau = \infty$ intersect and have the values $2\omega_\infty^{-2}$. For arbitrary τ at the point $\omega = \omega_*$ (2.5), (2.6) have the form

$$a_* = a_*^{(0)} \left(\frac{\varkappa_*^{\gamma}\nu^{-1} + \varkappa_*^{-\gamma}\nu + 2\cos\psi}{\varkappa_*^{\gamma}\nu^{-1} + \varkappa_*^{-\gamma}\nu - 2\cos\psi} \right)^{1/2} \tag{2.8}$$

$$\operatorname{tg}\varphi_{1*} = 2(\varkappa_*^{\gamma}\nu^{-1} - \varkappa_*^{-\gamma}\nu)\sin\psi \tag{2.9}$$

We shall study the behavior of the amplitude a_*, since (2.8) can be used to find the relaxational characteristics from experimental data. For fixed values of γ and $\xi = E_0/E_\infty$ the dependence of a_* on the relaxation time τ_ε is a symmetric peak, which under the condition $\varkappa_*^{\gamma} = \nu$ reaches the maximum

$$a_{*m} = a_*^{(0)} \operatorname{ctg} {}^1\!/_2\, \psi \tag{2.10}$$

For large and small values of τ the behavior of the amplitude a_* is defined respectively by the asymptotic formulas

$$\tau \gg 1, \qquad a_* \approx a_*^{(0)} (1 + 2\nu\varkappa_*^{-\gamma}\cos\psi) \tag{2.11}$$

$$\tau \ll 1, \qquad a^* \approx a_*^{(0)} (1 + 2\nu^{-1}\varkappa_*^{\gamma}\cos\psi) \tag{2.12}$$

Figure 1 shows the dependence of the quantity $b \equiv a_* - a_*^{(0)}$ on $\ln\tau_\varepsilon$ for $\nu_\varepsilon = 1$, ($\xi = 0$) $\omega_\infty = \sqrt{2}$. The numerals on the curves denote the values of the parameter γ.

We first investigate the dependence of the amplitude a_* on the divisibility parameter γ for fixed values of τ and ξ using (2.10) for the maximal value of a_{*m}. Then it is not difficult to see that a_{*m} increases monotonically from the value $a_*^{(0)}$ for $\gamma = 1$ to infinity as $\gamma \to 0$. The following asymptotic estimates hold

$$\gamma \to 1, \quad a_{*m} \approx a_*^{(0)} [1 + {}^1\!/_2\,\pi(1-\gamma)], \qquad \gamma \to 0, \quad a_{*m} \approx 8(\pi\omega_\infty^2\gamma)^{-1} \tag{2.13}$$

If $\varkappa_*^{\gamma} \neq \nu$, the behavior of a_* as $\gamma \to 1$ and $\gamma \to 0$ is defined by the expressions, respectively,

$$\gamma \to 1, \qquad a_* \approx a_*^{(0)} [1 + \pi(1-\gamma)\nu\varkappa_*(\nu^2 + \varkappa_*^2)^{-1}] \tag{2.14}$$

$$\gamma \to 0, \qquad a_* \approx a_*^{(0)} (1+\nu)^2 (1-\nu)^{-2} [1 + 2\gamma(\nu - \nu^{-1})^{-1} \ln\varkappa_*] \tag{2.15}$$

The amplitude a_* as a function of the degree of relaxation ξ, when τ and γ play the role of parameters, is defined in the region $\xi \in [0, 1]$ and for the same condition $\varkappa_*^{\gamma} = \nu$ (or in the variables τ_ε, ξ this is equivalent to $\xi = 1 - \varkappa_{*\varepsilon}^{\gamma}$) reaches the maximal value defined by (2.10). Hence we see that with variation of ξ the position of the peak on the $\ln\tau_\varepsilon$ axis changes and appears only for those values of the parameters τ_ε and γ which satisfy the condition $\varkappa_{*\varepsilon}^{\gamma} < 1$, rather than for any values of these parameters. The asymptotic behavior of the amplitude a_* as $\xi \to 0$ and $\xi \to 1$ is defined by the expressions, respectively,

$$\xi \to 0, \quad a_* \approx h[1 + 2\xi(\varkappa_{*\varepsilon}^{2\gamma} + \varkappa_{*\varepsilon}^{-2\gamma} - 2\cos 2\psi)^{-1}(\varkappa_{*\varepsilon}^{-\gamma} - \varkappa_{*\varepsilon}^{\gamma})\cos\psi] \tag{2.16}$$

$$h \equiv a_*^{(0)} (\varkappa_{*\varepsilon}^{\gamma} + \varkappa_{*\varepsilon}^{-\gamma} + 2\cos\psi)^{1/2} (\varkappa_{*\varepsilon}^{\gamma} + \varkappa_{*\varepsilon}^{-\gamma} - 2\cos\psi)^{-1/2}$$

$$\xi \to 1, \quad a_* \approx a_*^{0} [1 + 2\varkappa_{*\varepsilon}^{-\gamma}(1-\xi)\cos\psi] \tag{2.17}$$

Figure 2 shows the dependence of the quantity b on ξ for $\tau_\varepsilon = 1$, $\omega_\infty = \sqrt{2}$ for different γ, whose values are indicated by the numerals on the curves. We note that for $\xi = 1$ all the curves for a_* converge to the single point $a_* = a_*^{(0)}$ by virtue of the specific nature of the kernel (2.1) which describes the infinite creep process.

3. As the second example we consider the special case of the Rabotnov ϑ_γ function for $\nu = 1$. Then the relaxation and aftereffect kernels are written in the symmetric form

$$R(t) = \tau_\varepsilon^{-\gamma}\vartheta_\gamma(-1, \tau_\varepsilon, t), \qquad K(t) = \tau_\sigma^{-\gamma}\vartheta_\gamma(-1, \tau_\sigma, t) \tag{3.1}$$

In this case

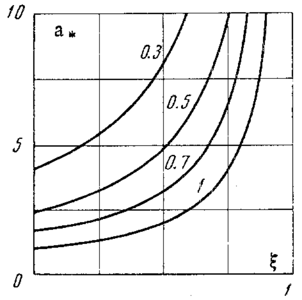

Fig. 3

$$A = \frac{\xi^{-1}\varkappa_\varepsilon^{\gamma} + \varkappa_\varepsilon^{-\gamma} + (1+\xi^{-1})\cos\psi}{\varkappa_\varepsilon^{\gamma} + \varkappa_\varepsilon^{-\gamma} + 2\cos\psi}, \qquad B = \frac{\nu_\sigma \sin\psi}{\varkappa_\varepsilon^{\gamma} + \varkappa_\varepsilon^{-\gamma} + 2\cos\psi} \tag{3.2}$$

$$C = \frac{\varkappa_\sigma^{\gamma} + \xi^{-1}\varkappa_\sigma^{-\gamma} + (1+\xi^{-1})\cos\psi}{\varkappa_\sigma^{\gamma} + \varkappa_\sigma^{-\gamma} + 2\cos\psi}, \qquad D = \frac{\nu_\sigma \sin\psi}{\varkappa_\sigma^{\gamma} + \varkappa_\sigma^{-\gamma} + 2\cos\psi} \tag{3.3}$$

In accordance with (1.6), (1.7), (1.16) we obtain for the amplitude, phase shift, and internal friction

$$a = \left(\frac{\varkappa_\varepsilon^{\gamma} + \varkappa_\varepsilon^{-\gamma} + 2\cos\psi}{\Omega_\infty^2\varkappa_\varepsilon^{\gamma} + \Omega_0^2\varkappa_\varepsilon^{-\gamma} + 2\Omega_\infty\Omega_0\cos\psi}\right)^{1/2}, \quad \Omega_0 \equiv \omega_0^2 - \omega^2 \tag{3.4}$$

$$\operatorname{tg}\varphi_1 = [\Omega_\infty\varkappa_\varepsilon^{\gamma} + \Omega_0\varkappa_\varepsilon^{-\gamma} + (\Omega_0 + \Omega_\infty)\cos\psi]^{-1}(\Omega_\infty - \Omega_0)\sin\psi \tag{3.5}$$

$$Q^{-1} = \operatorname{tg}\varphi_2 = [\varkappa_\varepsilon^{\gamma} + \xi\varkappa_\varepsilon^{-\gamma} + (1+\xi)\cos\psi]^{-1}\nu_\varepsilon \sin\psi \tag{3.6}$$

For $\gamma = 1$ all the relations (3.1)–(3.6) correspond to the standard linear body model, for which at the frequency

$$\omega_*^2 = {}^1/_2(\omega_\infty^2 + \omega_0^2) \tag{3.7}$$

the resonant amplitude a_* does not depend on the relaxation time τ_ε, i.e., all the amplitudes inersect at a single point, regardless of τ_ε [10]. For $\gamma \neq 1$ only the two resonant amplitudes corresponding to $\tau_\varepsilon = 0$ and $\tau_\varepsilon = \infty$ intersect at this point $\omega = \omega_*$. For arbitrary τ_ε at $\omega = \omega_*$, the amplitude a_* and the phase φ_{1*} are defined by the formulas

$$a_* = a_*^{(0)}\left(\frac{\varkappa_{*\varepsilon}^{\gamma} + \varkappa_{*\varepsilon}^{-\gamma} + 2\cos\psi}{\varkappa_{*\varepsilon}^{\gamma} + \varkappa_{*\varepsilon}^{-\gamma} - 2\cos\psi}\right)^{1/2}, \qquad a_*^{(0)} = \frac{2}{\omega_\infty^2 - \omega_0^2} \tag{3.8}$$

$$\operatorname{tg}\varphi_{1*} = 2(\varkappa_{*\varepsilon}^{\gamma} - \varkappa_{*\varepsilon}^{-\gamma})\sin\psi \tag{3.9}$$

We note that (3.9) is valid only when $\omega_\infty \neq \omega_0$. For $\omega_\infty = \omega_0$, i.e., in the absence of a modulus defect, it follows immediately from (3.5) that $\varphi_1 = 0$, which corresponds to the elastic solution.

Studies analogous to that of the Abel–Duffing kernel lead to the following results.

The function $a_* = f(\tau_\varepsilon)$ (γ and ξ are the parameters) forms a symmetric peak, which for the condition $\varkappa_{*\varepsilon} = 1$ reaches a maximal value

$$a_{*m} = a_*^{(0)}\operatorname{ctg}^{1/2}\psi \tag{3.10}$$

For large and small values of τ_ε the amplitude a_* approaches the value $a_*^{(0)}$, which is clearly seen from the following asymptotic estimates

$$\tau_\varepsilon \gg 1, \quad a_* \approx a_*^{(0)}(1 + 2\varkappa_\varepsilon^{-\gamma}\cos\psi) \tag{3.11}$$

$$\tau_\varepsilon \ll 1, \quad a_* \approx a_*^{(0)}(1 + 2\varkappa_\varepsilon^{\gamma}\cos\psi) \tag{3.12}$$

For the corresponding value of γ the curve of the function $a_* = f(\ln\tau_\varepsilon)$ is analogous to that shown in Fig. 1, and for $\xi = 0$ they coincide exactly.

For fixed values of τ_ε and ξ the amplitude a_* increases montonically with reduction of γ from the value $a_*^{(0)}$ for $\gamma = 1$ to infinity as $\gamma \to 0$. For example, for the condition $\varkappa_{*\varepsilon} = 1$, this is easily seen from (3.10) and for other values of $\varkappa_{*\varepsilon}$ the asymptotic behavior of a_* is defined by the formulas

$$\gamma \to 1, \quad a_* \approx a_*^{(0)}[1 + \pi(1-\gamma)(\varkappa_{*\varepsilon} + \varkappa_{*\varepsilon}^{-1})^{-1}] \tag{3.13}$$

$$\gamma \to 0, \quad a_* \approx 2a_*^{(0)}\gamma^{-1}[{}^1/_4\pi^2 + (\ln\varkappa_{*\varepsilon})^2]^{-1/2} \tag{3.14}$$

Finally, with change of the degree of relaxation ξ the amplitude a_* changes, increasing monotonically from the constant value for $\xi = 0$ to infinity as $\xi \to 1$.

This is not difficult to understand, since for $\omega_\infty^2 = \omega_0^2$ the elastic solution is obtained and the amplitude at resonance will naturally be infinite. As $\xi \to 0$ the asymptotic formula holds

$$\begin{gathered} a_* \approx a_\infty\{1 + \gamma\xi[1 + (\varkappa_\infty^{2\gamma} + \varkappa_\infty^{-2\gamma} - 2\cos 2\psi)^{-1}(\varkappa_\infty^{-\gamma} - \varkappa_\infty^{\gamma})\cos\psi]\} \\ a_\infty \equiv (\varkappa_\infty^{\gamma} + \varkappa_\infty^{-\gamma} + 2\cos\psi)^{1/2}(\varkappa_\infty^{\gamma} + \varkappa_\infty^{-\gamma} - 2\cos\psi)^{-1/2}, \quad \varkappa_\infty \equiv \omega_\infty\tau_\varepsilon/\sqrt{2} \end{gathered} \tag{3.15}$$

Figure 3 shows a_* as a function of the relaxation ratio ξ for various γ, whose values are noted on the curves ($\tau_\varepsilon = 1$, $\omega_\infty = \sqrt{2}$). The basic difference from the curves shown in Fig. 2 is that all the amplitudes a_* approach infinity as $\xi \to 1$, since they correspond to elastic resonance.

4. These results admit the possibility of their use for experimental applications. First, we note that the relaxation and aftereffect kernels discussed above are equivalent to quite definite relaxation (retardation) time distribution functions. For example, for the Abel–Duffing aftereffect kernel the corresponding distribution function is obtained most easily if we use the definition of the gamma function in terms of the Euler integral and make a change of the variables therein. As a result we can write the relation

$$t^{\gamma-1}\Gamma(1-\gamma) = \int_0^\infty \tau^{\gamma-2} e^{-t/\tau}\, d\tau \tag{4.1}$$

from which it follows that the retardation time distribution functions, equivalent to the Abel and Duffing kernels, have the forms respectively

$$f_A(\tau) = (\pi\tau_\sigma^\gamma)^{-1}\tau^{\gamma-1}\sin\pi\gamma\ ,\quad f_D(\tau) = [\tau_\sigma^\gamma\Gamma(1-\gamma)]^{-1}\tau^{\gamma-1} \tag{4.2}$$

The relaxation time distribution function for the Rabotnov kernel (2.2) can be obtained if we use the integral representation of the ϑ_γ function by the Mellin formula

$$\tau_\varepsilon^{-\gamma}\vartheta_\gamma(-\nu_\varepsilon, \tau_\varepsilon, t) = \frac{1}{2\pi i\nu_\varepsilon}\int_{c-i\infty}^{c+i\infty} \frac{\exp(pt)\, dp}{1+\nu_\varepsilon^{-1}(p\tau_\varepsilon)^\gamma} \tag{4.3}$$

Then, calculating the contour integral (4.3) by the methods of complex variable function theory, we obtain the relation

$$\tau_\varepsilon^{-\gamma}\vartheta_\gamma(-\nu_\varepsilon, \tau_\varepsilon, t) = \frac{\sin\pi\gamma}{\pi\nu_\varepsilon}\int_0^\infty \frac{\tau^{-2}\exp(-t/\tau)\, d\tau}{\nu_\varepsilon(\tau/\tau_\varepsilon)^\gamma + \nu_\varepsilon^{-1}(\tau/\tau_\varepsilon)^{-\gamma} + 2\cos\pi\gamma} \tag{4.4}$$

from which we see that the relaxation time distribution function is defined by the expression

$$f_R(\tau) = \frac{\sin\pi\gamma}{\pi\nu_\varepsilon\tau}\left[\nu_\varepsilon\left(\frac{\tau}{\tau_\varepsilon}\right)^\gamma + \nu_\varepsilon^{-1}\left(\frac{\tau}{\tau_\varepsilon}\right)^{-\gamma} + 2\cos\pi\gamma\right]^{-1} \tag{4.5}$$

For $\nu_\varepsilon = 1$ we obtain the distribution function corresponding to the kernel (3.1).

It follows from (4.2) and (4.5) that the parameter γ, defining the weak singularity of the hereditary functions, characterizes the "smearing" of the relaxation-retardation spectra. Therefore the experimental determination of γ is of definite interest. This is most easily done for the oscillator example if we study the temperature dependence of the amplitude a_* at the frequency $\omega = \omega_*$, assuming that the relaxation time τ_ε depends on the temperature T following the Arrhenius law

$$\tau_\varepsilon = \tau_{\varepsilon 0}\exp(-H/kT) \tag{4.6}$$

Here H is the activation energy of the relaxational process, k is the Boltzmann constant, and $\tau_{\varepsilon 0}$ is the frequency factor.

Then, knowing the maximal value of the amplitude a_{*m} and $a_*^{(0)}$, according to (2.10) and (3.10) we obtain

$$\pi\gamma = 4\ \mathrm{arc\ tg}\,(a_*^{(0)}/a_{*m}) \tag{4.7}$$

We note that the parameter γ determined in this way for the kernels (2.2) and (3.1) has different values, since in the first case γ corresponds to the background, while in the second case it corresponds to the relaxational peak of the internal friction Q^{-1} and the processes controlling these phenomena have a different nature. The values of γ obtained from the temperature dpendence of Q^{-1} for the background of certain pure metals (Cu, Al, Fe, Mo, W and others) lie in the range 0.2–0.3 [11]. The values of γ for the relaxational peaks are considerably higher for example for the grain boundary peak in polycrystalline aluminum ≈ 0.7 [12].

Thus, study of the temperature dependence of the oscillator amplitude a_* in the steady-state regime makes it possible to determine the parameter γ by the same method for fundamentally different phenomena.

Another important characteristic of the relaxational process is the activation energy H, which in the present method, regardless of whether it is related with the peak or the background, is defined by the same formula:

$$H = kT_1T_2(T_2 - T_1)^{-1}\ln(\omega_{*2}^{2}\,\omega_{*1}^{-2}) \qquad (4.8)$$

For an unchanged relaxation ratio the variation of the frequency ω_* is achieved by varying the mass. The frequency factor for the background is found from the formula

$$\ln\tau_{\varepsilon 0} = \ln\nu_\varepsilon - {}^1\!/_2\,[\ln(\omega_{*1}\omega_{*2}) + Hk^{-1}(T_1^{-1} + T_2^{-1})] \qquad (4.9)$$

and for the peak we must set $\nu_\varepsilon = 1$ in (4.9).

Thus, study of the resonant amplitude of the hereditarily elastic oscillator at the frequency $\omega = \omega_*$ makes it possible to find all the parameters of the relaxational spectrum.

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